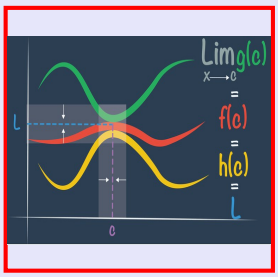
Math 261

Fall 2023

Lecture 12



Feb 19-8:47 AM

Class QZ 7

Class room for exam : G5-118

Use ε and δ definition to prove $\lim_{x\to 6}\left(\frac{1}{6}x+4\right)=5$

$f(x)=\frac{1}{6}x+4$

$L=5$

$a=6$

1) verify the limit

$\lim_{x\to 6}\left(\frac{1}{6}x+4\right)=\frac{1}{6}(6)+4=1+4=5$ ✓

For $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|\frac{1}{6}x+4-5\right|<\varepsilon$

$\left|\frac{1}{6}x-1\right|<\varepsilon$

Multiply by 6

$6\left|\frac{1}{6}x-1\right|<6\varepsilon$

$|x-6|<6\varepsilon$

$|x-6|<\delta$

Choose

$\boxed{\delta=6\varepsilon}$ ✓✓

Sep 14-11:20 AM

For any function $f(x)$, the first derivative of it is $f'(x)$, F-prime of x, and can be found by evaluating the following limit if exists

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: Find $f'(x)$ for $f(x) = 3x^2 - 5$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 5 - (3x^2 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-5} \cancel{-3x^2} \cancel{+5}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} = \lim_{h \to 0} (6x + 3h)$$

$$= \boxed{6x}$$

Sep 18-10:30 AM

Find $f'(x)$ for $f(x) = \frac{1}{x}$ using the limit def.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

LCD $= x(x+h)$

$$= \lim_{h \to 0} \frac{x\cancel{(x+h)} \cdot \frac{1}{\cancel{x+h}} - \cancel{x}(x+h) \cdot \frac{1}{\cancel{x}}}{x(x+h) \cdot h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{x(x+h) \cdot h} = \lim_{h \to 0} \frac{-\cancel{h}}{x(x+h) \cdot \cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \boxed{\frac{-1}{x^2}}$$

$f(x) = \frac{1}{x} \Rightarrow$ Domain $(-\infty, 0) \cup (0, \infty)$

$f'(x) = \frac{-1}{x^2} \Rightarrow$ " $(-\infty, 0) \cup (0, \infty)$

Sep 18-10:37 AM

Use limit def. of $f'(x)$ to find $f'(x)$ for $f(x)=\sqrt{x}$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\overset{A}{(\sqrt{x+h}}\overset{+B}{+\sqrt{x})}\overset{A}{(\sqrt{x+h}}\overset{-B}{-\sqrt{x})}}{(\sqrt{x+h}+\sqrt{x})\cdot h}=\lim_{h\to 0}\frac{\overset{A^2}{(\sqrt{x+h})^2}\overset{-B^2}{-(\sqrt{x})^2}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{\cancel{x}+\overset{1}{\cancel{h}}-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\boxed{\frac{1}{2\sqrt{x}}}$$

$f(x)=\sqrt{x}$ → Domain $[0,\infty)$

$f'(x)=\frac{1}{2\sqrt{x}}$ → Domain $(0,\infty)$

Sep 18-10:43 AM

Given $f(x)=x^2-4x$

1) find $f(2)$ $\quad f(2)=2^2-4(2)=4-8=\boxed{-4}$

2) find $f'(x)$ using limit def.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-4(x+h)-x^2+4x}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{-4x}-4h\cancel{-x^2}\cancel{+4x}}{h}=\lim_{h\to 0}\frac{\cancel{h}(2x+h-4)}{\cancel{h}}$$

$$=\lim_{h\to 0}(2x+h-4)=2x+0-4=\boxed{2x-4}$$

3) find $f'(2)=2(2)-4=\boxed{0}$

Sep 18-10:50 AM

Use ε and δ definition to prove $\lim_{x \to 0} x^3 = 0$.

$f(x) = x^3$

$L = 0$

$a = 0$

verify the limit

$\lim_{x \to 0} x^3 = 0^3 = 0$ ✓

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.

$|x^3 - 0| < \varepsilon$ " $|x - 0| < \delta$

$|x^3| < \varepsilon$ $|x| < \delta$

$\sqrt[3]{|x|^3} < \sqrt[3]{\varepsilon}$ Take cube root of both sides

$|x| < \sqrt[3]{\varepsilon}$

$\boxed{\text{Pick } \delta = \sqrt[3]{\varepsilon}}$

Sep 18-10:57 AM

Use ε and δ definition to prove $\lim_{x \to 5}(x^2 + 4x) = 45$.

$f(x) = x^2 + 4x$

$L = 45$

$a = 5$

verify the limit

$\lim_{x \to 5}(x^2 + 4x) = 5^2 + 4(5) = 25 + 20 = \boxed{45}$ ✓

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 + 4x - 45| < \varepsilon$ " $|x - 5| < \delta$

$|(x+9)(x-5)| < \varepsilon$ " $|x - 5| < \delta$

$|x+9|\,|x-5| < \varepsilon$ $\delta = \frac{\varepsilon}{C}$

if $|x+9| < C$, then $C|x-5| < \varepsilon \to |x-5| < \frac{\varepsilon}{C}$

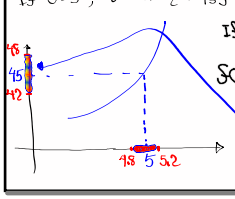
If we choose $\delta \le 1$, then $|x-5| < 1$

$-1 < x - 5 < 1 \xrightarrow{+5} 4 < x < 6 \xrightarrow{+9} 13 < x+9 < 15$

we conclude $|x+9| < 15$ So $C = 15$

$\delta = \min\left\{1, \frac{\varepsilon}{15}\right\}$

If $\varepsilon = 3$, $\delta = \min\left\{1, \frac{3}{15}\right\} = \min\left\{1, \frac{1}{5}\right\} = \frac{1}{5} = .2$



If $x = 5.1$

$f(5.1) = (5.1)^2 + 4(5.1)$

$= 26.01 + 20.4$

$= \boxed{46.41}$

Sep 18-11:04 AM

Use ε and δ definition to prove $\lim_{x \to 2} x^3 = 8$

$f(x) = x^3$ For every $\varepsilon > 0$, there is a $\delta > 0$

$L = 8$ ✓ Such that

$a = 2$ $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^3 - 8| < \varepsilon$ whenever $|x - 2| < \delta$

Factor

$|(x-2)(x^2 + 2x + 4)| < \varepsilon$ whenever $|x - 2| < \delta$

Keep Bound $\delta = \frac{\varepsilon}{C}$

If $|x^2 + 2x + 4| < C$, then $|(x-2) \cdot C| < \varepsilon \to |x - 2| < \frac{\varepsilon}{C}$

If we wish $\delta \le 1$, then $|x - 2| < 1$

$-1 < x - 2 < 1$ $\xrightarrow{+2}$ $1 < x < 3 \to |x| < 3$

$|x| < 3 \to |x^2| < 9$, $|2x| < 6$

$|x^2 + 2x + 4| < 9 + 6 + 4 < 19$

So $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$

Sep 18-11:17 AM

Use ε and δ definition to prove $\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$

$f(x) = \frac{1}{x}$ Verify the limit

$L = 2$ $\lim_{x \to 1/2} \frac{1}{x} = \frac{1}{1/2} = 2$ ✓

$a = \frac{1}{2}$

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{1}{x} - 2\right| < \varepsilon$ " $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{1 - 2x}{x}\right| < \varepsilon$ " $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{-2(x - \frac{1}{2})}{x}\right| < \varepsilon$ " $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{-2}{x}\left(x - \frac{1}{2}\right)\right| < \varepsilon$ Pick $\delta = \frac{\varepsilon}{C}$

$\left|\frac{2}{x}\right| \left|x - \frac{1}{2}\right| < \varepsilon$

If $\left|\frac{2}{x}\right| < C$, then $C\left|x - \frac{1}{2}\right| < \varepsilon$, $\left|x - \frac{1}{2}\right| < \frac{\varepsilon}{C}$

Sep 18-11:28 AM